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Information and Entropy

Spring 2006

Issued: April 20, 2006, 1:00 PM

Quiz

Due: April 20, 2006, 2:00 PM

Note: Please make sure you follow this instructions

- Write your name at the top of each page in the space provided → worth 1% of the grade.
- You may bring and use any notes you want on one 8 1/2 x 11 sheet of paper. You may also use a hand calculator, but the quiz is designed so that you do not need one (see the table of logarithms on the last page which you may tear off and use for reference).
- Put your answers on this sheet and turn it in along with any calculations you do on other pieces of paper. Be sure to put your name on all pages you turn in.

Problem 1: (24%)

Your company manufactures AND gates that have the probability diagram of the form:

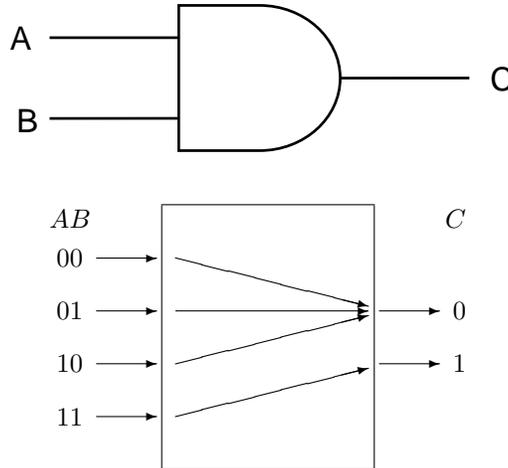


Figure 1: AND gate

If input probabilities of A and B are independent and each has equal probability for 0 and 1, $P(A = 0) = 1/2$ and $P(B = 0) = 1/2$,

a. Find $P(C = 0)$ and $P(C = 1)$.

$$P(C = 0) = \underline{\hspace{2cm}} \quad P(C = 1) = \underline{\hspace{2cm}}$$

- b. Find the gate's input information I , output information J , loss L , noise N and mutual information M .

$$I = \text{_____} \quad J = \text{_____} \quad L = \text{_____} \quad N = \text{_____} \quad M = \text{_____}$$

Your boss is concerned that the gate has so much loss and asks to have it redesigned. Your colleague Ben Bitdiddle decides to convert it to a two-input, two-output gate with outputs C and D , where C is A AND B as before and $D = A$.

- c. Show the transition probabilities for Ben's gate in the figure below (just draw those with nonzero probabilities.)

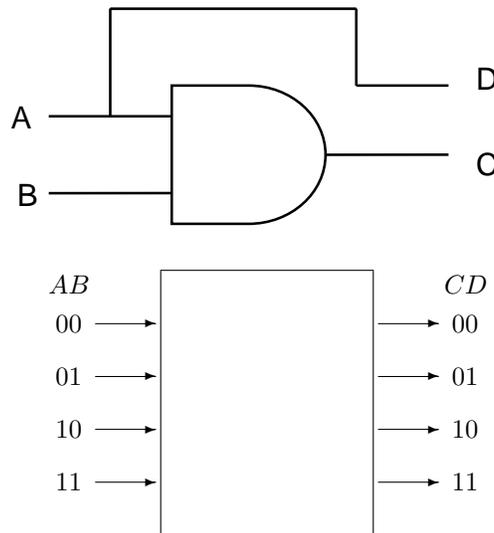


Figure 2: Ben's gate

- d. Calculate the following probabilities:

$$P(CD = 00) = \text{_____} \quad P(CD = 01) = \text{_____} \quad P(CD = 10) = \text{_____} \quad P(CD = 11) = \text{_____}$$

- e. Find the input information I , output information J , loss L , noise N and mutual information M for Ben's gate.

$$I = \text{_____} \quad J = \text{_____} \quad L = \text{_____} \quad N = \text{_____} \quad M = \text{_____}$$

- f. Is Ben's gate reversible?

Yes _____ No _____

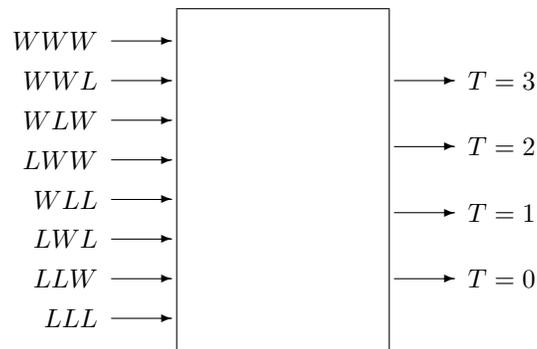
If it is, show its inverse. If not, identify at least two inputs that produce the same output.

Problem 2: (35%)

The MIT Solar Electric Vehicle Team (SEVT) wants to take their car on a tour of three solar-car competitions in the summer of 2006: the Formula Sun Grand Prix, the American Solar Challenge, and the World Solar Challenge (after they solve a halting problem on this touring machine). With typical MIT arrogance, Jason, the SEVT expert on information and entropy, estimates the probability of SEVT winning a trophy in each of the independent events at 0.5.

Jason knows *The Tech* (the MIT student newspaper) and other newspapers will care about the total number T of trophies won (0, 1, 2, or 3). Therefore he calculates the probabilities of each of these possible outcomes and devises a process flow model for this calculation, where the input is the eight possible win/loss patterns.

- a. Fill in this process model box and give the probabilities for the four possible values of T .



$$P(T = 0) = \underline{\hspace{2cm}} \quad P(T = 1) = \underline{\hspace{2cm}} \quad P(T = 2) = \underline{\hspace{2cm}} \quad P(T = 3) = \underline{\hspace{2cm}}$$

- b. Find the information at the input of this process I_{in} , the output information I_{out} , the noise N , loss L , and mutual information M , all in bits.

$$I_{in} = \underline{\hspace{2cm}} \quad I_{out} = \underline{\hspace{2cm}} \quad L = \underline{\hspace{2cm}} \quad N = \underline{\hspace{2cm}} \quad M = \underline{\hspace{2cm}}$$

After the last race (held in Australia), Jason wants to send the value of T to newspapers, who will not remember the results of the prior competitions. However, communicating from the remote location is expensive. His colleagues have told him that he will need to send two bits to encode the four possible outcomes. The cost is \$0.13 for one bit, or \$0.25 for two. On a limited budget, Jason does not want to pay even the two-bit rate.

- c. Help him out by devising a code for the four outcomes that has an average message length shorter than two bits, and say whether the expected cost is greater or less than \$0.25.

Code:

T = 0 : _____ T = 1 : _____ T = 2 : _____ T = 3 : _____

Average message length in bits: _____ Average message cost in dollars: _____

Problem 3: (40%)

On your vacation you find yourself in a gambling casino that caters to nerds. One of the games catches your attention because you wonder about the probabilities. In this game, you pay a nickel (5 cents) to play, and then a coin is chosen at random from a big jar and given to you. The jar contains pennies (worth 1 cent), nickels (5 cents), and quarters (25 cents). This game, unlike slot machines, has some payout each time you play. You notice that the jar is so large that the probabilities P , N , and Q of the three coins (penny, nickel, and quarter) being selected, remain the same all day long.

- a. In the absence of other information, what set of probabilities P , N , and Q express your knowledge without any additional assumptions?

$$P \text{ _____ } N \text{ _____ } Q \text{ _____}$$

- b. You ask the owner of the casino about the probabilities P , N , and Q but he says the only thing he knows is that the average payout is 4 cents per play. Write a constraint equation that relates P , N , and Q that incorporates this new information.

Constraint Equation: _____

- c. What values of P , N , and Q are compatible with this information and also the fact that P , N , and Q must add up to 1?

$$P_{min} \text{ _____ } P_{max} \text{ _____ } N_{min} \text{ _____ } N_{max} \text{ _____ } Q_{min} \text{ _____ } Q_{max} \text{ _____}$$

- d. You know that the probabilities that are consistent with your knowledge but have no additional assumptions is found by expressing your uncertainty as a function of one of the probabilities and then finding where its maximum occurs. Write this formula as a function of any one of the three probabilities P , N , and Q . (You do not need to find this maximum point.)

Equation for the Entropy: _____

Without a calculator you can't find the maximum of this expression, so instead you guess that it is about 0.5 bits.

Then you realize that the owner need not stock the jar using maximum entropy. He could set the probabilities so as to maximize the number of quarters paid out, while still keeping an average payout of 4 cents (this choice would be exciting to non-nerd customers hoping to strike it rich.)

- e. With this strategy, what are P , N , Q , and the entropy (to two decimal places)?

P _____ N _____ Q _____ Entropy = _____

- f. Was your earlier quick guess for the entropy (when you thought the Principle of Maximum Entropy was used by the owner) a good one? Explain your answer.

Logarithm and Entropy Table

This page is provided so that you may rip it off the quiz to use as a separate reference table. In Table 1, the entropy $S = p \log(1/p) + (1 - p) \log_2(1/(1 - p))$.

p	1/8	1/5	1/4	3/10	1/3	3/8	2/5	1/2	3/5	5/8	2/3	7/10	3/4	4/5	7/8
$\log_2(\frac{1}{p})$	3.00	2.32	2.00	1.74	1.58	1.42	1.32	1.00	0.74	0.68	0.58	0.51	0.42	0.32	0.18
S	0.54	0.72	0.81	0.88	0.92	0.95	0.97	1.00	0.97	0.95	0.92	0.88	0.81	0.72	0.54

Table 1: Table of logarithms in base 2 and entropy in bits