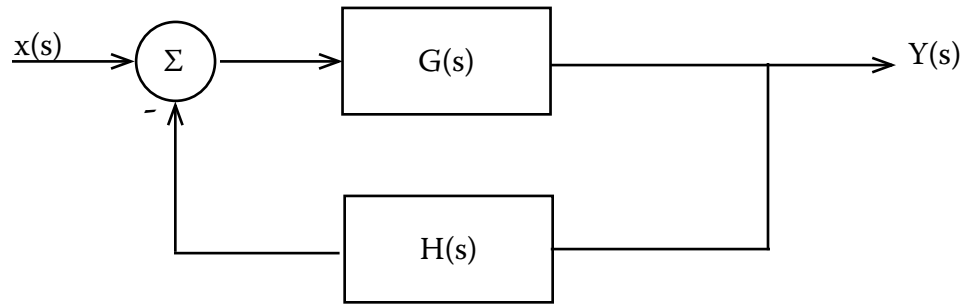


# 6.302 Feedback Systems

Recitation 17: Black's Formula Revisited, and Lead Compensation

Prof. Joel L. Dawson

By now, applying Black's Formula to a feedback system is almost a reflex:



This formula actually lends itself rather naturally to graphical interpretation. Consider three different regions for the magnitude of  $L(s)$  [=  $G(s)H(s)$ ].

1)  $|L(s)| \gg 1$

Here  $|G(s)H(s)| \gg 1 \rightarrow |G(s)| \gg \frac{1}{|H(s)|}$

and the closed-loop transfer function is well approximated by

$$\frac{Y(s)}{x(s)} \approx \frac{1}{H(s)}$$

2)  $|L(s)| \ll 1$

Here  $|G(s)H(s)| \ll 1 \rightarrow |G(s)| \ll \frac{1}{|H(s)|}$

and the closed-loop transfer function is well approximated by

$$\frac{Y(s)}{x(s)} \approx G(s)$$

3)  $|L(s)| = 1$

This is how things are at the loop crossover frequency. At this point:

$$|G(s)H(s)| = 1 \rightarrow |G(s)| = \frac{1}{|H(s)|}$$

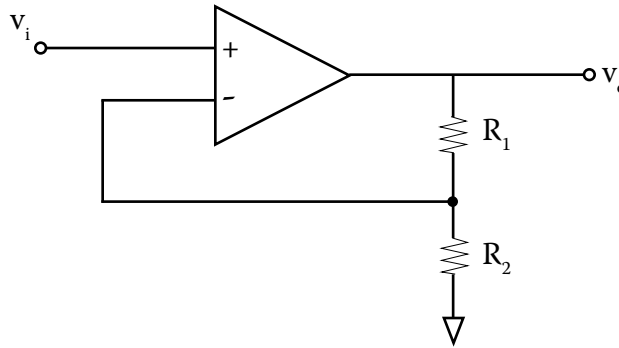
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The graphical interpretation goes as follows. To quickly graph the closed-loop response of a feedback system, we just overlay a plot of  $|G(s)|$  on top of a plot of  $|H(s)|^{-1}$ . The closed-loop curve is given by tracing the lower curve at all frequencies! And...loop crossover occurs at the intersection of the two curves.

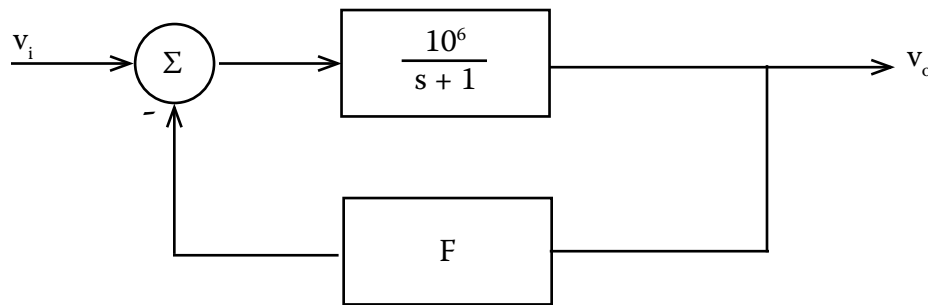
EXAMPLE: Dominant-pole compensated op-amps.



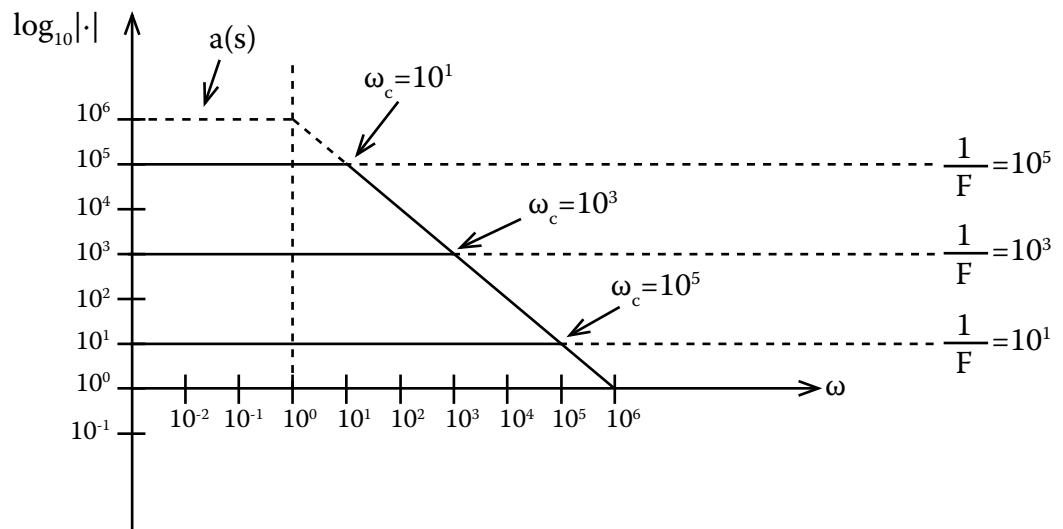
$$a(s) = \frac{10^6}{s + 1}$$

$$F = \frac{R_2}{R_2 + R_1}$$

Block Diagram:



Bode Plots:



Notice: Gain-Bandwidth Product =  $\frac{1}{F} \omega_c = \text{constant for } \frac{1}{|F|} < 10^6$ .

This is not a fundamental law, but a reflection of the first order roll-off of the dominant-pole compensated op-amp.

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## Wrap-up of Lag Compensation

In the last recitation we covered lag compensation. In general, it is used in one of two ways.

- 1) Decrease  $|L(s)|$  at high frequencies, while leaving low-frequency behavior unchanged:

$$G_c(s) = \frac{\tau s + 1}{\alpha \tau s + 1}$$

- 2) Increase  $|L(s)|$  at low frequencies, while leaving high-frequency behavior unchanged:

$$G_c(s) = \alpha \left[ \frac{\tau s + 1}{\alpha \tau s + 1} \right]$$

Remember, it is the 3-degrees of freedom (pole placement, zero placement, and gain selection) that gave us an advantage over pure proportional control (pure gain selection).

## Lead Compensation

Lead Compensation reverses the order of the pole and zero. The general form here is

$$G_c(s) = k_L \left[ \frac{\alpha \tau s + 1}{\tau s + 1} \right]$$

And we tend to use lead compensation when we want one of three things:

- 1) By placing the lead zero at crossover (and the pole well above crossover), we can pick up  $45^\circ$  of phase margin without changing the crossover frequency.
- 2) By choosing a crossover frequency that falls at the geometric mean of the pole and zero, we can maximize our phase margin increase.
- 3) By using the lead zero to cancel a system pole, we effectively move that pole a factor of  $\alpha$  higher in frequency.

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Let's see a couple of examples of lead compensation. Suppose that we've decided to let  $k_L = 1$ , so

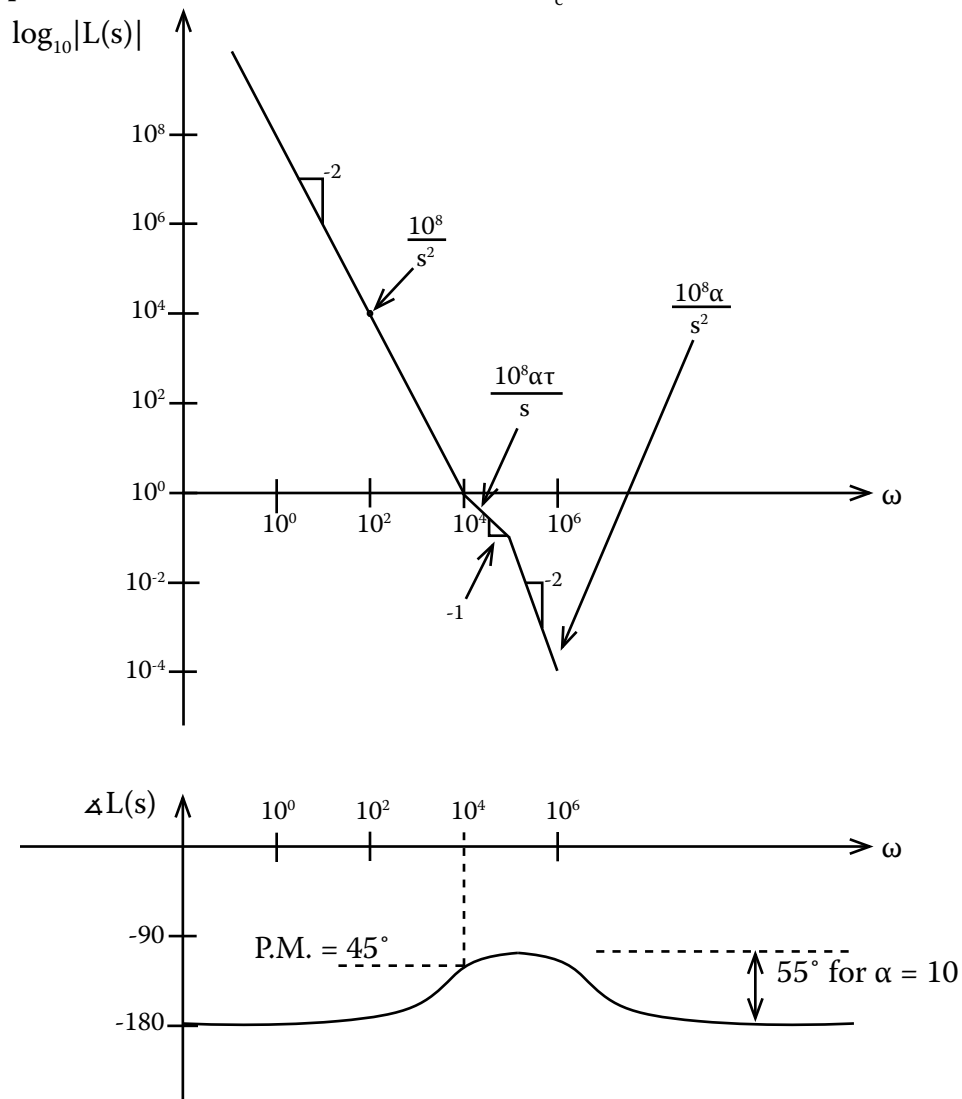
$$G_c(s) = \frac{(\alpha\tau s + 1)}{(\tau s + 1)}$$

and the uncompensated system is

$$G(s) = \frac{10^8}{s^2}$$

for reasons of noise rejection, we will restrict ourselves to  $\alpha \leq 10$ . (This is not a hard and fast rule. We just want to put a flag in your mind about getting  $90^\circ$  phase bumps out of this technique.)

First, we'll place the lead zero at the old crossover,  $\omega_c = 10^4$ .



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Notice how, in the asymptotic approximation, the magnitude curve progresses as follows:

$$\frac{10^8}{s^2} \longrightarrow \frac{10^8 \alpha \tau s}{s^2} \longrightarrow \frac{10^8 \alpha \tau s}{s^2 \tau s}$$

$\omega$  increases

Useful picture for navigating around Bode Plots.

What about placing lead compensator such that we get the maximum phase bump to occur @ loop crossover?

In the vicinity of crossover, the loop transmission is going to look like

$$\frac{10^8}{s^2} \cdot \alpha \tau s = \frac{10^8 \alpha \tau}{s}$$

We require that the new crossover frequency be the geometric mean of the lead zero and lead pole frequency. That is

$$\omega_c = \sqrt{\frac{1}{\alpha \tau} \frac{1}{\tau}} = \frac{\alpha^{-1/2}}{\tau}$$

This allows us to solve for  $\tau$ :

$$L(j\omega_c) = 1 \rightarrow \frac{10^8 \alpha \tau}{\omega_c} = \frac{10^8 \alpha \tau}{\alpha^{-1/2} / \tau} = 1$$

$$10^8 \alpha^{3/2} \tau^2 = 1$$

$$\tau^2 = 10^{-8} \alpha^{-3/2}$$


$$\tau = 10^{-4} \alpha^{-3/4}$$

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Notice that we had to have the crossover frequency move from its original location. This happened because we have not used all of our degrees of freedom:

$$G_c(s) = k_L \left[ \frac{\alpha\tau s + 1}{\tau s + 1} \right]$$


If we use both  $k_L$  and  $\tau$ , we can get maximum phase bump and keep crossover the same.

Next time: Minor Loop Compensation

