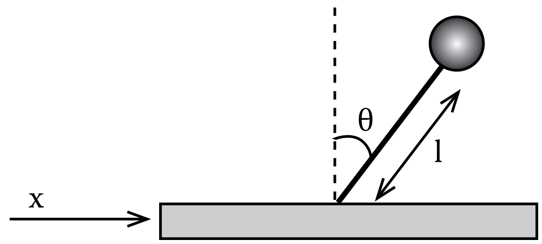
6.302 Feedback Systems

Recitation 21: Inverted Pendulum
Prof. Joel L. Dawson

Today we're going to talk about one of the most amazing lecture demos EVER: the inverted pendulum.

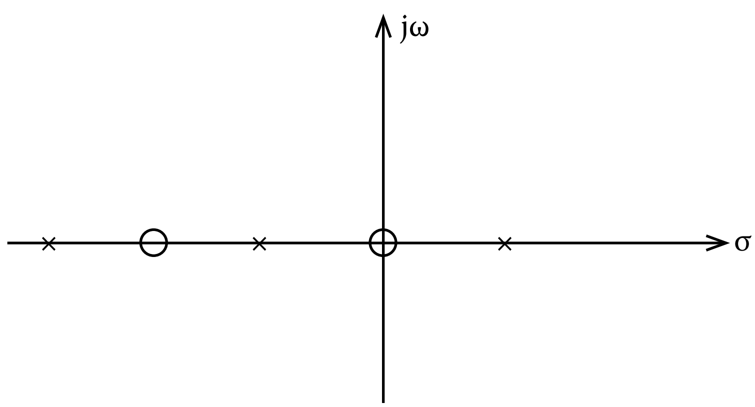


It's amazing because it looks an awful lot like a machine that thinks. It doesn't, of course, but seeing this thing in action really tugs at your intuition. Or, at any rate, my intuition.

As a prelude to understanding this system, let's consider the following example of how positive feedback can be a good thing.

CLASS EXERCISE

Say you have an uncompensated system with a pole/zero diagram that looks like this:

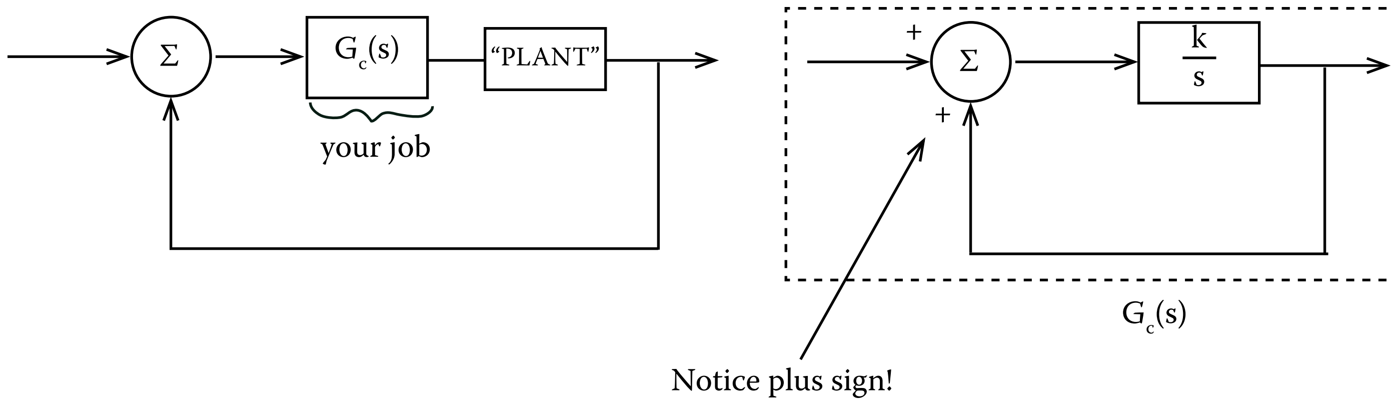


1) Can this system be made stable for finite k?

2) If not, add one singularity in order to make things better.

So where does the positive feedback come in? It comes in when you try to actually realize a right-half-plane pole.

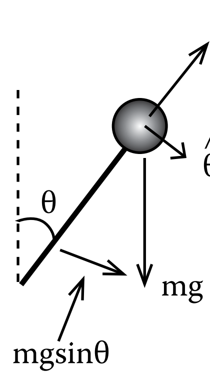
Consider:



Interesting minor loop.

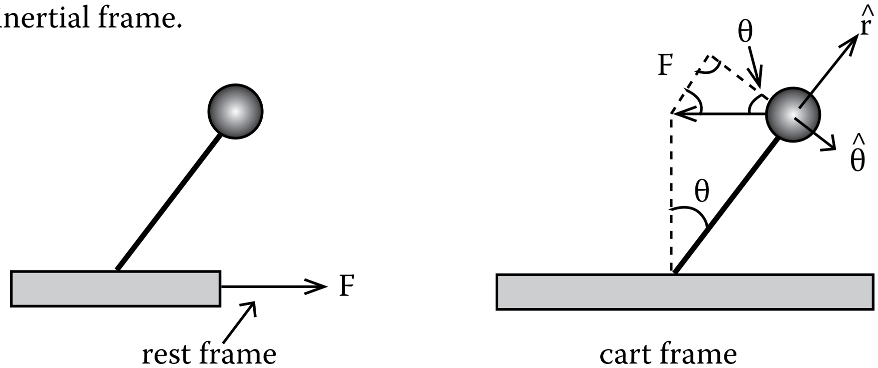
Now let's look at the inverted pendulum. First we need a mathematical description of the plant.

<u>Gravity</u>



<u>Torque</u>: $T_G = lF_\theta = mgl\sin\theta$

<u>Force of the cart</u>: This one's tricky. We want to be in the frame of reference of the cart, which is a non-inertial frame.



$T_c = lF_\theta$
$= -Fl\cos\theta$

But $F = m\ddot{x}$, where m is the mass of the ball.

$$\hookrightarrow T_c = -m\ddot{x}l\cos\theta$$

Moment of inertia of ball on a massless rod: ml^2. So the equation of motion is

$$\Sigma T = \cancel{m}\cancel{l}^2\ddot{\theta} = \cancel{m}g\cancel{l}\sin\theta - \cancel{m}\ddot{x}\cancel{l}\cos\theta$$

$$\ddot{\theta} = \frac{g}{l}\sin\theta - \frac{\ddot{x}}{l}\cos\theta$$

Linearize: $$\ddot{\theta} = \frac{g}{l}\theta - \frac{\ddot{x}}{l}$$

(Gives eqn of motion for small θ. That's where we hope we are!)

$$s^2\Theta(s) = \frac{g}{l}\Theta(s) - \frac{1}{l}s^2X(s)$$

and get the plant:

$$\boxed{\frac{\Theta(s)}{X(s)} = \frac{-s^2/g}{(\tau_c s + 1)(\tau_c s - 1)}}$$

Tells us how cart position affects pendulum angle. The setup for the demo has a pulley system that converts shaft rotation of a motor directly to translational position of the cart. If we drive the motor with a voltage,

$$\frac{X(s)}{V(s)} = \frac{k_m}{s(\tau_m s + 1)}$$

We've now done all the modeling that we need to do. Our first crack at a control system, then, might be proportional control. ⇒

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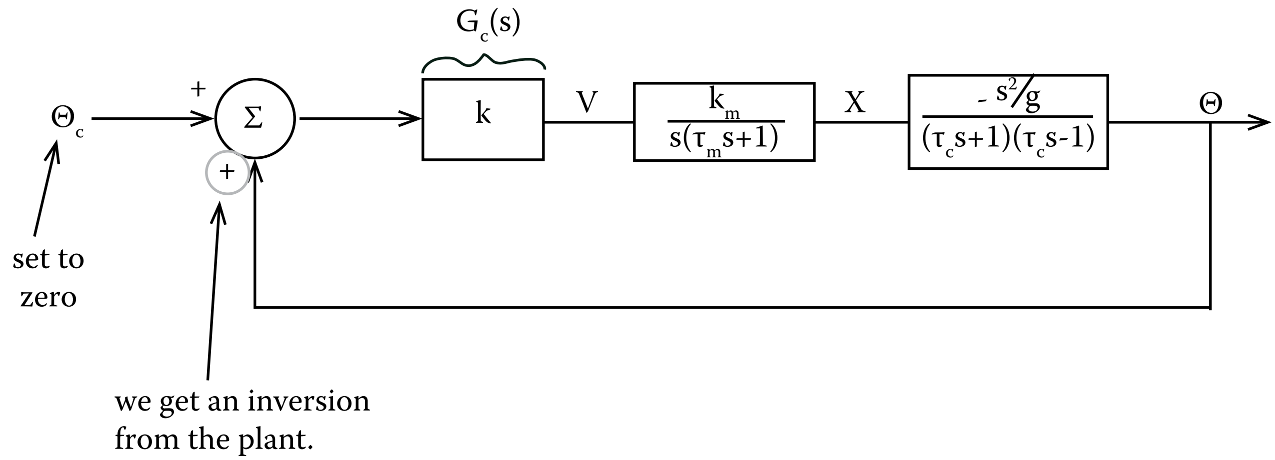
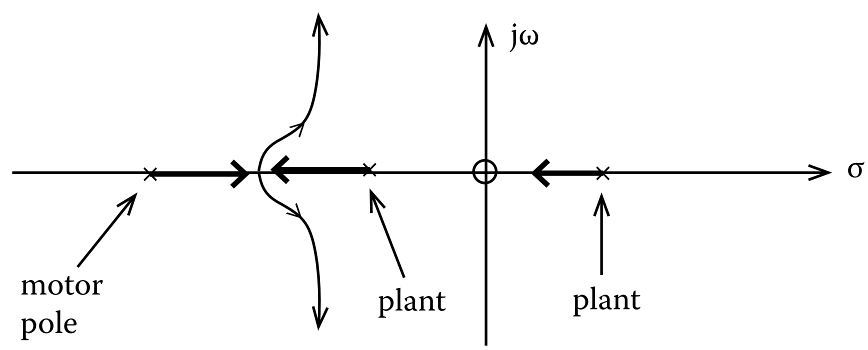


Root Locus

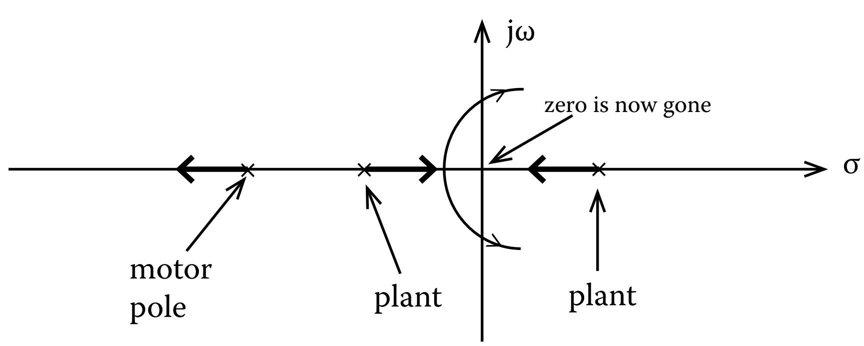


Unstable for all k.

What's happening here? An angle error $\Theta_c - \Theta \neq 0$ sets a cart <u>velocity</u>. A constant error $\Rightarrow$ constant cart velocity, and no force exerted to right the pendulum.

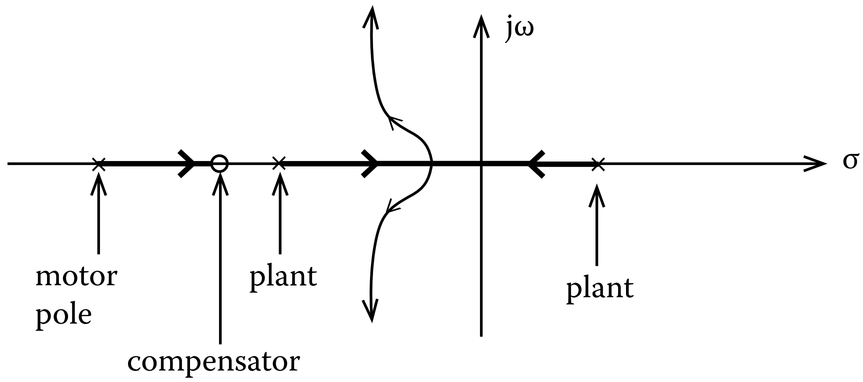
What we really want is for $\Theta_c \neq \Theta$ to cause an acceleration. This is done by putting a pole at the origin in $G_c(s)$.

$$\rightarrow G_c(s) = \frac{k}{s}$$

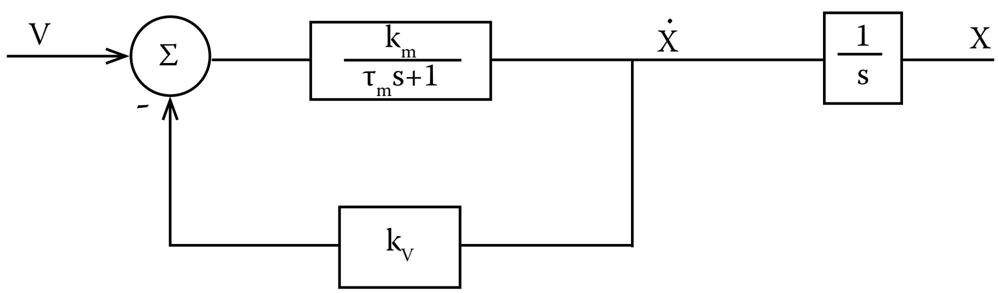


We can do better. By adding a zero, p-z becomes two again and we get the asymptotes to be ± 180°.

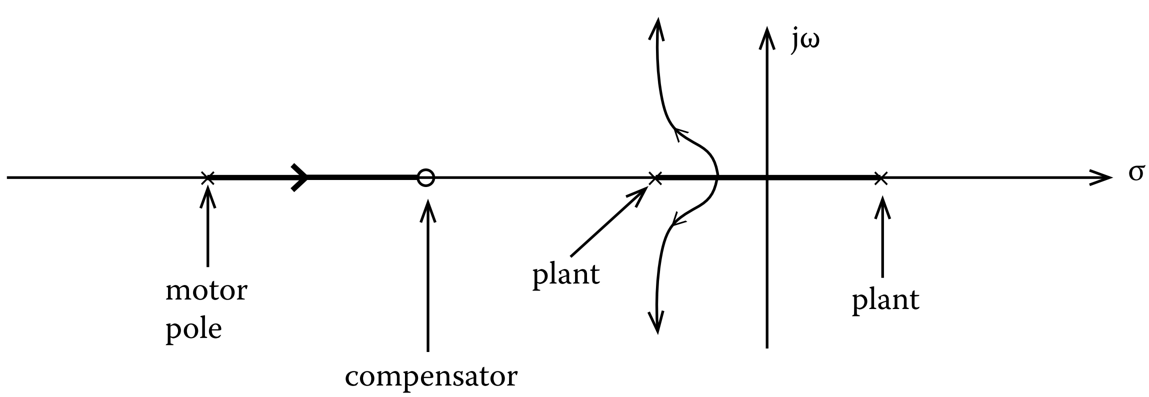
$$G_c(s) = k\,\frac{(\tau_k s + 1)}{\tau_k s}$$



Can we improve the stability margins? Perhaps we could by adding more poles and zeros. But minor loop feedback provides us with an easy fix. Use velocity feedback to speed up motor response:



Now root locus looks like:



We've made a lot of progress with this, but one more problem remains. See the text section 5.6 for details, and notice how positive feedback gave us the degree of freedom to really wrap this system up.