

6.302 Feedback Systems

Recitation 7: Root Locus

Prof. Joel L. Dawson

To start with, let's make sure we're clear on exactly what we mean by the words "root locus plot." Webster can help us with this:

ROOT: "A number that reduces an equation to an identity when it is substituted for one variable."

The equation that we care about is $1 + L(s) = 0$, and the identity that it reduces to is $0 = 0$ when s is chosen to be a root. Roots of this equation are the closed-loop poles of the feedback system.

LOCUS: "The set of all points whose location is determined by stated conditions."

The "stated conditions" here are that $1 + kL_o(s) = 0$ for some value of k , and the "points" whose locations matter to us are points in the s -plane.

Put them together:

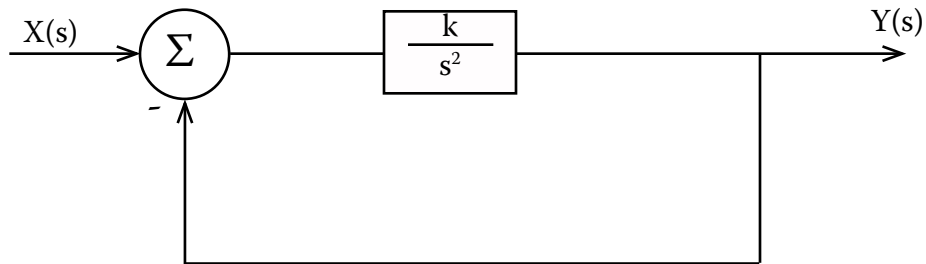
ROOT LOCUS: The set of all points in the s -plane that satisfy the equation $1 + kL_o(s) = 0$ for some value of k .

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Can begin to construct such a plot for a simple system:



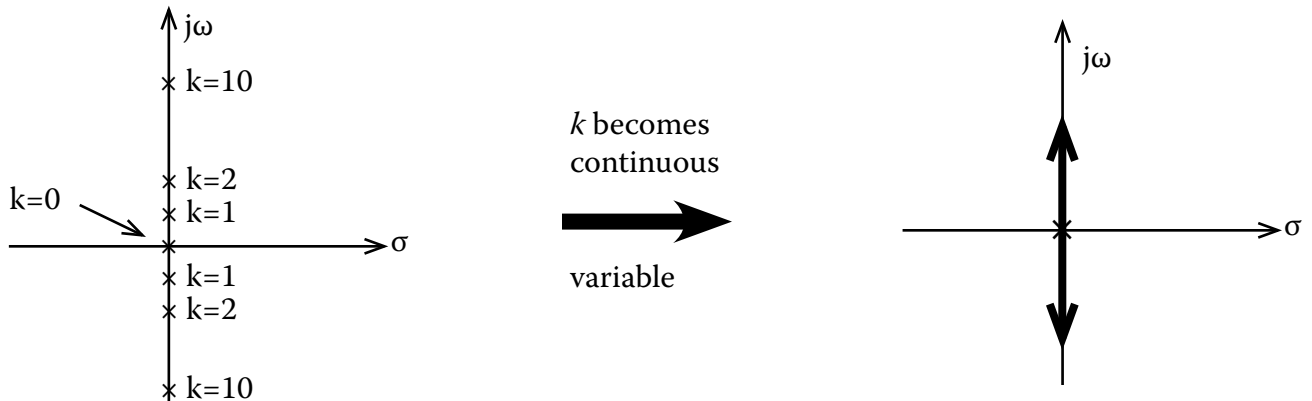
Where are the closed-loop poles?

$$\frac{Y(s)}{x(s)} = \frac{k/s^2}{1 + k/s^2} = \frac{k}{s^2 + k} \quad \left. \vphantom{\frac{Y(s)}{x(s)}} \right\} \begin{array}{l} s^2 = -k \\ s = \pm j \sqrt{k} \end{array}$$

Evidently, the answer depends on k . We can build a table:

k	Pole Locations
0	0, 0 (two poles @ origin)
1	$\pm j$
2	$\pm j \sqrt{2}$
10	$\pm j \sqrt{10}$

How would this look on a parameterized pole-zero diagram?



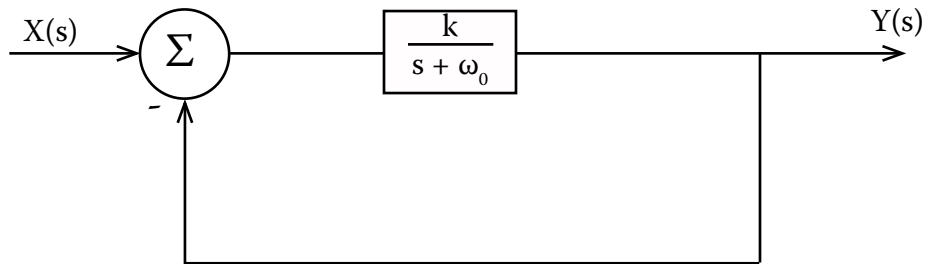
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CLASS EXERCISE

Draw the root locus plot for the following system:



Now, on to the rules of root locus plotting. In the last two examples, we could do a plot just based on algebra. But what about when the order of the characteristic equation is too high for that? We use the rules, which all derive from

$$1 + kL_o(s) = 0$$
$$kL_o(s) = -1$$

This equation implied two things:

- ① $|kL_o(s)| = 1 \Rightarrow$ magnitude condition
- ② $\angle kL_o(s) = -180^\circ$

or, for positive k

$\rightarrow \angle L_o(s) = -180^\circ \Rightarrow$ angle condition

An astonishing number of rules can be derived from these simple conditions.

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RULE #1: The number of branches, which are the paths of the closed-loop poles, is equal to the number of open-loop poles.

Characteristic equation:

$$P(s) = 1 + kL_o(s) = 0$$

$$\text{Let } L_o(s) = \frac{n(s)}{d(s)}$$

$$P(s) = 1 + k \frac{n(s)}{d(s)} = 0$$

$$\hookrightarrow d(s) + kn(s) = 0$$

$L_o(s)$ is a physical system \rightarrow has at least as many poles as zeros $\rightarrow d(s)$ is of order greater than or equal to $n(s)$ \rightarrow the order of $d(s)$ is the order of $d(s) + kn(s)$.

In other words, closing a loop around something doesn't alter the number of poles.

RULE #2: The branches start at the open-loop poles, and end at open-loop zeros. (In addition to the z open-loop zeros in the loop transmission, there are $p-z$ open-loop zeros at infinity.)

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To prove this rule (#2), we rely on the magnitude condition:

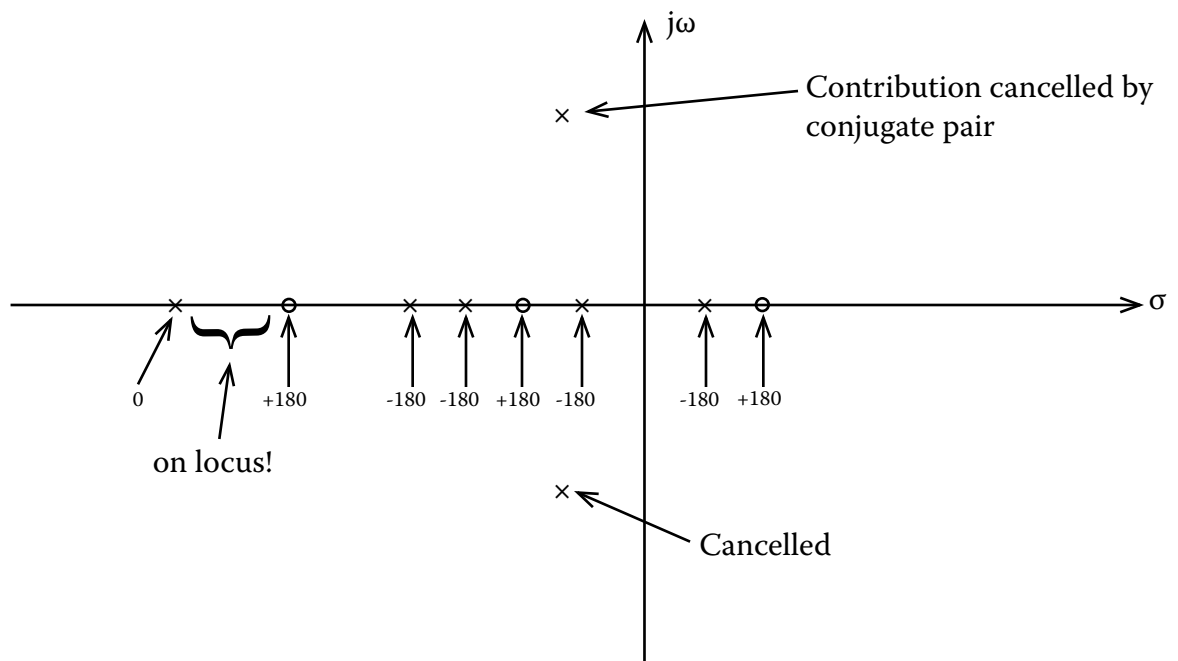
$$|kL_o(s)| = 1$$

We say “branches start out” when we’re talking about small values of k , and talk of “branches ending” when k is LARGE. The thought experiment here is that we’re beginning with k small, cranking it higher and higher, and watching where the poles go.

- For LARGE k , $L_o(s)$ must be small. For infinite k , $L_o(s)$ must be zero.
- For small k , $L_o(s)$ must be LARGE. For zero k , $L_o(s)$ must be infinite.

RULE #3: Branches of the root locus lie on the real axis to the left of an odd number of real poles and zeros. Complex-conjugate pairs of poles do not count, since on the real axis they contribute no net angle.

Angle condition: $\angle L_o(s) = (2n + 1) \cdot 180^\circ$



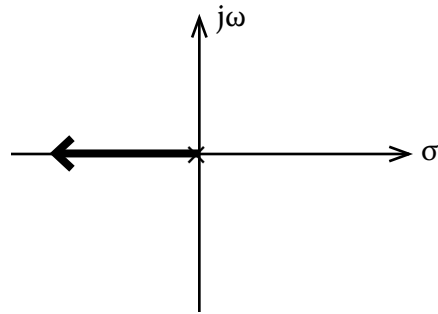
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Examples using what we have so far:

$$L(s) = \frac{k}{s}$$

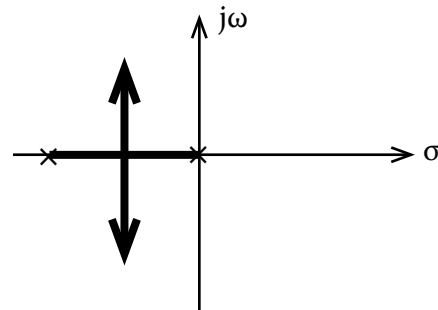


Rule 1: → 1 branch

Rule 2: → Goes off to infinity

Rule 3: → Locus on real axis to the left of one pole

$$L(s) = \frac{k}{s(s+1)}$$



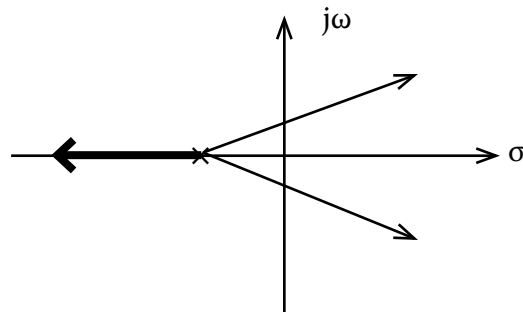
Rule 1: → 2 branches

Rule 2: → 2 poles go off to infinity

Rule 3: → Locus on real axis lies to the left of one pole

Rule 4: → (We don't know this one yet.)

$$L(s) = \frac{k}{(s+1)^3}$$



Rule 1: → 3 branches

Rule 2: → 3 poles go off to infinity

Rule 3: → Locus on real axis lies to the left of three poles

+ more rules

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RULE #4: If a branch on the real axis lies between a pair of poles, the root locus must break away from the real axis somewhere between the poles. Similarly, if a branch on the real axis lies between a pair of zeros, there must be an entry point between that pair of zeros.

Back to characteristic equation:

$$1 + k \frac{n(s)}{d(s)} = 0$$

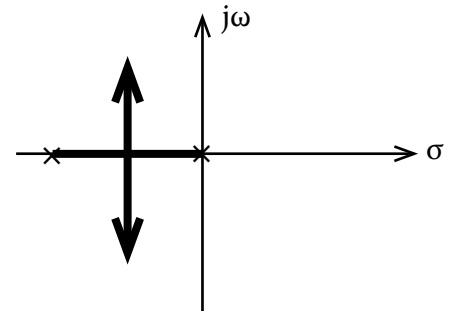
$$d(s) + kn(s) = 0$$

→ polynomial with real coefficients

- ⇒ Poles occur only in conjugate pairs
- ⇒ Root locus symmetric about real axis

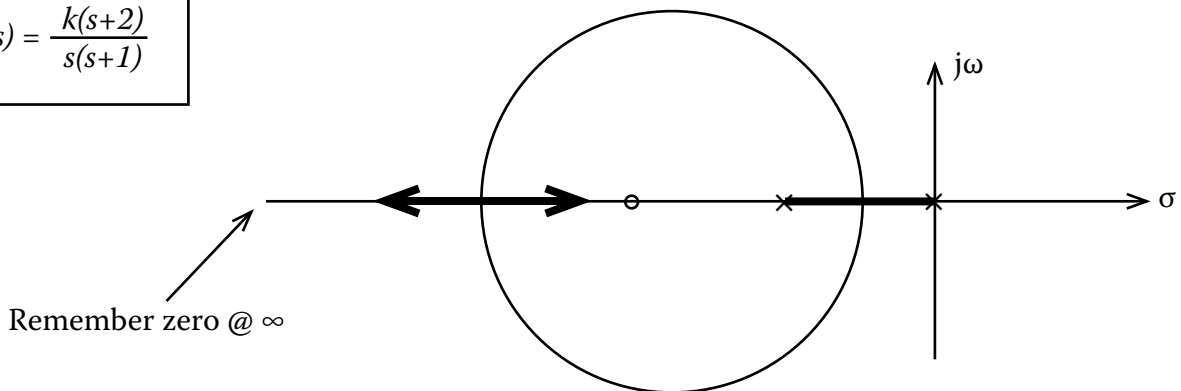
EXAMPLE:

$$L(s) = \frac{k}{s(s+1)}$$



(Note: Poles could not simply pass through one another and remain on real axis.)

$$L(s) = \frac{k(s+2)}{s(s+1)}$$



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RULE #5: As k gets very large, $P-Z$ branches go off to infinity (rule 2). These branches approach asymptotes as angles to the real axis of

$$\alpha_n = \frac{(2n + 1) 180^\circ}{P - Z}$$

Where $n = 0 \dots (P-Z-1)$ and the centroid of these asymptotes is on the real axis at

$$\sigma_a = \frac{\sum p_i - \sum z_i}{P - Z}$$

(offered w/out proof)

RULE #6: The departure angles of the branches from an m^{th} order pole on the real axis are

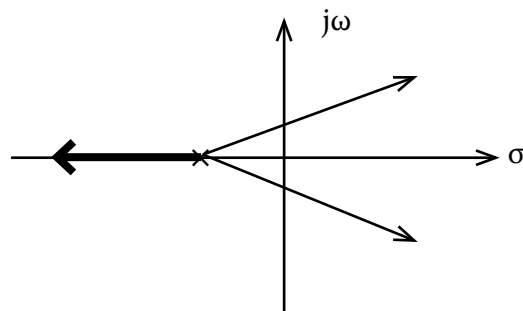
$$\delta_n = \frac{(2n + 1) 180^\circ}{m}$$

If the m^{th} order pole is to the left of an even number of poles and zeros. If to the left of an odd number, the departure angles are

$$\delta_n = \frac{2n180^\circ}{m}$$

KEY: In the vicinity of these poles, the angle contributed by all other singularities looks constant.

$$L(s) = \frac{k}{(s+1)^3}$$



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EXAMPE: $L(s) = \frac{ks}{(s+1)^3}$

Rule 1: → 3 branches

Rule 2: → 2 branches go off to infinity, one heads toward zero

Rule 3: → Real axis between poles and zero is on the locus

Rule 4: → N/A

Rule 5: → -asymptotes are at $90^\circ, 270^\circ$

-centroid at $\frac{-3-0}{2} = -1.5$

Rule 6: → Departure angles determined by $\delta_n = \frac{2n180^\circ}{m} = \{0, 120^\circ, 240^\circ\}$

